PMM U.S.S.R.,Vol.52,No.3,pp.405-408,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

ASYMPTOTIC ANALYSIS OF THE STABILITY OF A CYLINDRICAL VISCOELASTIC SHELL UNDER THE ACTION OF A LONGITUDINAL PERIODIC LOAD*

L.KH. BELEN'KAYA

Stability of the rectilinear form of a viscoelastic, orthotropic cylindrical shell acted upon by a longitudinal periodic load is considered in the case of high-frequency modulation and near the resonance frequencies. The boundary conditions at the ends of the shell allow a periodic continuation over the spatial variables, so that the problem can be reduced to a system of ordinary integrodifferential equations with periodic coefficients.

The problem of the stability of a cylindrical shell acted upon by a longitudinal periodic load was studied in /1/, where the Bolotin method was used for the exponential relaxation kernel to derive an approximate formula for the critical modulation frequency near the principal resonance at low modulation amplitudes and low viscosities. The study of the stability of the rectilinear form of the cylindrical shell in question was carried out numerically, using the method of continued fractions over a wide a range of parameters of the system**. (**Belen'kaya L.Kh. Numerical study of the stability of an orthotropic, viscoelastic cylindrical shell acted upon by a longitudinal periodic load. Rostov-on-Don, 1985. Dep. in VINITI, 7898-84, 11.12.84.)

Below we obtain an asymptotic formula for the boundary of stability in the case when $\omega\to\infty$ and the wave numbers are fixed. In the case of fractional exponential relaxation kernels the critical load and neutral oscillations are sought in the form of series in fractional powers of the parameter ε $(\varepsilon=1/\omega)$. For the differential equations in the case when the relaxation kernel has no singularities at the zero, and for the integrodifferential equations, the asymptotic expansions are constructed in integral powers of ε. It is shown that at high modulation frequencies the critical value of the load is close to its stationary value.

Further, the Lyapunov-Schmidt method is used to study the behaviour of the system in the case when the frequency ω is close to the resonance frequency ω_k $(\omega_k=2\omega_*/k,\ k=1,2,3,\ldots;\ \omega_*$ is the natural frequency of oscillation) and the coefficients of viscosity are small. It is shown that if the mean value of the load $\langle\varphi\rangle\neq 0$, then at the higher-order resonances $(k=2,3,\ldots)$ it strongly shifts the frequency of natural oscillations and a stable state exists near the k-th resonance $(k=2,3,\ldots)$ when the viscosity is low, i.e. the instability near the higher-order resonances $(k=2,3,\ldots)$ is quenched by the viscosity.

If on the other hand $\langle\varphi\rangle=0$, then a state of instability exists near the k-th resonance $(k=2,3,\ldots)$.

1. Formulation of the problem.

A longitudinal periodic load $\varphi(\omega\tau)=\beta(1+\mu\cos\omega\tau)$ where β is the mean axial load pressure, μ, ω are the modulation amplitude and frequency and d, δ denote the dimensionless length and thickness of the shell, acts on an orthotropic, viscoelastic cylindrical shell. The equations of motion of the shell given in /1/ are used. The ends of the shell can move freely in the axial direction, but not in the radial direction, so that the initial system reduces to a system of ordinary integrodifferential equations with periodic coefficients

$$\begin{aligned}
&W^{\cdot\cdot}-\beta\lambda^2(2\pi)^{-1}(1+\mu\cos\omega\tau)W-\lambda^2 f+{}^1/_2(nY-\lambda X)-\\
&\quad {}^1/_2 nQ_{44}^*(V_3Y)(\tau)+{}^1/_2\lambda Q_{55}^*(V_2X)(\tau)=0\\
&a_{11}W+a_{12}X+a_{13}Y+a_{14}(V_1W)(\tau)+a_{15}'(V_1X)(\tau)+a_{15}''(V_2X)(\tau)+\\
&\quad a_{16}(V_1Y)(\tau)=0\\
&a_{21}W+a_{22}X+a_{23}Y+a_{24}(V_1W)(\tau)+a_{25}(V_1X)(\tau)+a_{26}'(V_1Y)(\tau)+\\
&\quad a_{26}''(V_3Y)(\tau)=0\\
&a_{31}f+a_{32}W+a_{33}(V_1W)(\tau)+a_{34}(V_1f)(\tau)=0\\
&(V_iu)(\tau)\equiv\int_{\tau_0}^{\tau}K_i(\tau-\theta)u(\theta)\,d\theta;\quad \lambda=\frac{\pi l}{d},\quad i=1,2,3
\end{aligned}\tag{1.1}$$

Here $K_i(\theta)$ are the relaxation kernels, W, X, Y, f are the amplitudes of the displacement, of rotation vector and stress function, n and l are the azimuthal and axial quantum numbers, and $n=0,1,2,\ldots;\ l=1,2,\ldots$. The dimensionless coefficients a_{ij}, Q_{44}^*, Q_{55}^* depend on the material constants and quantum numbers n and l (see the footnote on the previous page).

In order to study the stability of the rectilinear form, we shall seek non-zero solutions of system (1.1) at $\tau_0=-\infty$, of the form

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$$(W, X, Y, f) = e^{\sigma\tau}(W_1, X_1, Y_1, f_1) \tag{1.2}$$

where W_1, X_1, Y_1, f_1 are time periodic functions $(p = 2\pi/\omega)$; σ is a complex parameter. Following the theory of ordinary differential equations, we shall call the solutions (1.2) the Floquet solutions. We shall also call the set of values of σ for which system (1.1) has Floquet solutions, the spectrum of stability of problem (1.1). By analogy with the first Lyapunov method for ordinary differential equations we shall assume that the shell is asymptotically stable, provided that the spectrum of stability of system (1.1) lies in the left half-plane $(\mathrm{Re}\,\sigma < 0)$, and unstable if at least one value σ_0 is such that $\mathrm{Re}\,\sigma_0 > 0$.

We shall consider the loss of stability of the rectilinear form of the shell related to the appearance of periodic oscillations with period p or $2p$. We shall denote by $\beta_k{}^1$ the critical values of the load β corresponding to the p-periodic perturbations of the system (1.1) $(\tau_0 = -\infty)$, and by $\beta_k{}^2$ corresponding to $2p$-periodic perturbations (n and l are fixed, $k = 1, 2, \ldots N$).

2. Limits of stability for large ω. We will obtain an asymptotic formula for the boundary of stability in the case when the modulation frequency $\omega \to \infty$. We put $\omega\tau = t$ in the system (1.1) and introduce a small parameter $\varepsilon = 1/\omega$. We impose on W the normalizing condition $\langle W \rangle = 1$ and write $W = 1 + w$, $\langle w \rangle = 0$. Let us consider a specific case, when the kernels K_i are fractionally exponential

$$K_i(t) = A_i e^{-\gamma t} t^{\alpha-1}, \quad A_i \geqslant 0$$
$$0 < \alpha < 1, \quad \gamma > 0 \tag{2.1}$$

We reduce system (1.1) to the form

$$\begin{gathered} w^{\cdot\cdot} - \varepsilon^2\lambda^2(2\pi)^{-1}\beta(1 + \mu\cos t)(1 + w) - \varepsilon^2\lambda^2 f + {}^1/_2\varepsilon^2(nY - \lambda X) - \\ {}^1/_2\varepsilon^2\delta Q_{44}{}^* n A_2 (V_\varepsilon Y)(t) + {}^1/_2\varepsilon^2\delta\lambda Q_{55}{}^* A_3 (V_\varepsilon X)(t) = 0 \\ a_{11}(1 + w) + a_{12}X + a_{13}Y + \delta a_{14}A_1(V_\varepsilon w)(t) + \delta(a_{15}{}'A_1 + \\ a_{15}{}''A_2)(V_\varepsilon X)(t) + \delta A_1 a_{16}(V_\varepsilon Y)(t) + a_{14}A_1\overline{K} = 0 \\ a_{21}(1 + w) + a_{22}X + a_{23}Y + \delta a_{24}A_1(V_\varepsilon w)(t) + \delta a_{25}A_1(V_\varepsilon X)(t) + \\ \delta(a_{26}{}'A_1 + a_{26}{}''A_2)(V_\varepsilon Y)(t) + a_{24}A_1\overline{K} = 0 \\ a_{31}f + (1 + w)a_{32} + a_{33}A_1\overline{K} + a_{33}A_1\delta(V_\varepsilon w)(t) + \delta a_{34}A_1(V_\varepsilon f)(t) = \\ 0; \quad \delta \equiv \varepsilon^\alpha, \quad \overline{K} = \Gamma(\alpha)/\gamma^\alpha \end{gathered} \tag{2.2}$$

We shall seek the solutions of the system (2.2) and the critical load, in the form of a series in powers of ε, δ. It can be shown that if the kernels K_i have no singularities at zero, then the expansions can be carried out in integral powers of ε:

$$(w, X, Y, f) \sim \sum_{k,l=0}^{\infty}(w_{kl}, x_{kl}, y_{kl}, f_{kl})\varepsilon^k\delta^l \tag{2.3}$$
$$\beta_1{}^1 \sim \sum_{k,l=0}^{\infty}\beta_{kl}\varepsilon^k\delta^l$$

(We know (e.g. /2/) that in the case of differential equations an analogous expansion is carried out in integral powers of ε). We shall not expand the kernel $K(\theta)$ in a series ε, δ. Substituting (2.3) into system (2.2) and equating terms of like powers in ε, δ, we obtain a sequence of systems from which we can determine, one after the other, $w_{kl}, x_{kl}, y_{kl}, f_{kl}, \beta_{kl}$. The necessary and sufficient condition for the existence of a 2π-periodic solution of the resulting systems is, that the following condition holds:

$$\langle G_{kl} \rangle = 0 \tag{2.4}$$

where G_{kl} is the right-hand side of the differential equation of the systems.

Using this condition, we obtain the asymptotic formula for the critical load $\beta_1{}^1$ (the quantum numbers n and l are fixed) as $\omega \to \infty$

$$\beta_1{}^1 = \beta_{00} + \beta_{20}\varepsilon^2 + \beta_{40}\varepsilon^4 + \beta_{41}\varepsilon^4\delta + O(\varepsilon^6)$$
$$\beta_{20} = \lambda^2\mu^2\beta_{00}{}^2/(4\pi) \tag{2.5}$$

Here β_{00} is the critical load for the stationary problem $(\mu = 0)$, and the coefficients β_{41}, β_{40} depend on the parameters of the system (the expressions themselves are not given because of their length).

We see from (2.5) that at high modulation frequencies ω the value of the critical load β is close to the value of the critical load for the stationary case $(\mu = 0)$.

Calculations were carried out for the following values of the moduli of elasticity: $A_{11} = 2.089 \times 10^{10}$ N/m^2, $A_{12} = 0.276 \times 10^{10}$ N/m^2, $A_{22} = 22.75 \times 10^{10}$ N/m^2, and $A_{44} = A_{55} = A_{66} = 0.795 \times 10^{10}$ N/m^2, and $A_{44}{}^* = A_{55}{}^* = A_{66}{}^* = 0.1 \cdot A_{66}$. The parameters of the relaxation kernels are $\alpha_i = 0.25$, $\gamma_i = 0.05$. (The kernels $K_i(t)$ are given by the relation (2.1)).

At high values of the modulation frequency ω the critical load obtained numerically with help of continued fractions, reaches the asymptotic value calculated according to formula

(2.5). The asymptotic formula gives good agreement with the value of the critical load determined numerically already for $\omega=3$, $A_1=1/2$, $A_2=A_3=0$, $d=2$ when $(n,l)=(9,10)$ (we note that when $\omega=3$, then there exist several points $(9,l)$ at which the critical load attains its minimum value). For $\mu\in(0;7]$ the critical load is identical with the asymptotic value (2.5) with an error not exceeding 3.4%.

For $\omega=10$, $A_1=1/2$, $A_2=A_3=0$, $(n,l)=(9,10)$ the critical load obtained is numerically equal to the asymptotic value, with an error not exceeding 0.03% for $\mu\in(0;1]$. Two terms of the formula (2.5) produce an error not exceeding 0.3%.

At high modulation frequencies an increase in the amplitude μ exerts a stabilizing influence (the critical load increases as μ increases).

We note that the coefficient β_{20} of the asymptotic expansion (2.5) can be found from the numerical results (see the earlier footnote)

$$\beta_{20}=\lim_{\omega\to\infty}\omega^2(\beta_*^1-\beta_{00}) \tag{2.6}$$

Here β_*^1 is the critical load obtained numerically for fixed n, l, μ, A_i, ω; β_{00} is the critical load for $\mu=0$. We will consider the case $A_1=1/2$, $A_2=A_3=0$, $\mu=1/2$, $(n,l)=(9,10)$; with these parameters the stationary load $\beta_{00}=0.1688$. The coefficient β_{20} obtained from Eq.(2.6) is identical with the asymptotic value $\lambda^2\mu^2\beta_{00}^2/(4\pi)$ for $\omega=10$ with an error of 3.1%, and for $\omega=20$ with an error of 0.78%. If the modulation amplitude is $\mu=0.2$, then the coefficient β_{20} calculated from formula (2.6) will be identical with the asymptotic value (2.5) at $\omega=7$ with the error of 1.5%.

3. The behaviour of the system near the resonance frequencies. Let us investigate the behaviour of system (1.1) in the case when the shall material has low ductility and the modulation frequency ω is close to one of the resonance frequencies $\omega_k=2\omega_*/k$ $(k=1,2,\ldots$, ω_* is one of the natural frequencies of the shell defined by the quantum numbers (n,l)).

Let us construct the neutral curves near every resonance frequency. To do this, it is convenient to introduce into system (1.1) a formally small parameter ε and replace the kernels $K_i(\tau)$ by εK_i. We can assume that the parameter ε is responsible for the inherent properties of the shell. We further introduce into our discussion the detuning of the k-th resonance, assuming that

$$\alpha\equiv\omega_*^2-\omega^2k^2/4 \tag{3.1}$$

Perhaps it would be more natural to fix ω_* and vary ω, but the substitution adopted here, to which we can pass by an appropriate change of time, is more convenient.

Let us rewrite system (1.1) when $\tau_0=-\infty$, taking into account the remarks made above and using (3.1). We shall study the stability of this system for small $\varepsilon, \alpha, \beta$.

When $\varepsilon=0, \beta=0$, the system is stable. If ω does not coincide with any of the resonance frequencies ω_k, then the system remains stable also for small ε, β and the critical load tends, as $\varepsilon\to0$, to a non-zero limit β_0. In this case the region of stability in the (β,ε) plane contains an open semicircle: $\beta^2+\varepsilon^2<\beta_0^2$, $\varepsilon>0$. If on the other hand the frequency ω tends to one of the resonance frequencies $(\alpha\to0)$ and $\varepsilon\to0$, then the critical load will tend to zero. In this case the points of the upper semicircle around the zero in the β,ε plane lying in some curvilinear sector, will appear in the region of instability.

We shall use the Lyapunov-Schmidt method is determine the neutral curves near the resonances, and construct the corresponding branching equation. When $\varepsilon=\beta=\alpha=0$, the system has the solution

$$W_0=Ae^{ik\omega\tau/2}+A^*e^{-ik\omega\tau/2},\quad f_0=-a_{32}W_0/a_{31}$$
$$X_0=\chi_0W_0,\quad Y_0=U_0W_0$$

(A, A^* are unknown constants).

We shall seek a periodic solution with frequency $\omega m/2$ $(m=1,2)$ of the system in question, for small ε,α,β in the form

$$W=W_0+\overline{W},\ X=X_0+\overline{X},\quad Y=Y_0+\overline{Y},\ f=f_0+\overline{f} \tag{3.2}$$

Here we have

$$\langle\overline{W}e^{\pm ik\omega\tau/2}\rangle=0$$
$$(\overline{W},\overline{X},\overline{Y},\overline{f})(\tau+4\pi/m)=(\overline{W},\overline{X},\overline{Y},\overline{f})(\tau) \tag{3.3}$$

We shall seek the unknown functions $\overline{W}, \overline{X}, \overline{Y}, \overline{f}$ in the form of series in powers of $\alpha, \varepsilon, \beta$

$$(\overline{W},\overline{X},\overline{Y},\overline{f})=\sum(w_{klm},x_{klm},y_{klm},f_{klm})$$
$$\alpha^k\varepsilon^l\beta^m,\quad k^2+m^2+l^2>0 \tag{3.4}$$

Substituting (3.2) and (3.3) into the system in question, taking into account (3.4), and writing the conditions of solvability of the resulting system, we obtain the branching equation which defines, for fixed and sufficiently small ε, in the α,β plane, a neutral curve separating the regions of stability and instability.

The neutral curve is given, with the accuracy up to second-order infinitesimals, by the equation

$$\beta_{1,2} \approx [-p_0(\mathrm{Re}\,E_1\varepsilon - \alpha) \pm \Delta^{1/2}]/g_{002} \tag{3.5}$$
$$g_{002} = \lambda^2 (2\pi)^{-1}(p_0^2 - |p_k|^2)$$
$$\Delta = (\mathrm{Im}\,E_1)^2 (|p_k|^2 - p_0^2)\varepsilon^2 + |p_k|^2 (\alpha - \varepsilon\,\mathrm{Re}\,E_1)^2$$
$$p_k = \langle \Phi e^{-ik\omega\tau}\rangle, \quad \Phi = \sum_{j=-\infty}^{\infty} p_j e^{ij\omega\tau}$$

$\Phi(\tau)$ is the periodic load with period $p = 2\pi/\omega$, acting upon the shell, and E_1 is a coefficient depending on the inherent properties of the shell.

We have the following possibilities:

1°. If $p_0^2 < |p_k|^2$ $(g_{002} < 0)$, then the branching equation always has a solution (3.5) and we have instability near the k-th resonance $(k = 1, 2, \ldots)$ for small $\alpha, \varepsilon, \beta$.

2°. If $p_0^2 > |p_k|^2$ $(g_{002} > 0)$, then, provided that $\Delta < 0$, the branching equation has no solutions and we have stability near the k-th resonance $(k = 1, 2, \ldots)$ for small $\alpha, \varepsilon, \beta$. When $\Delta \geqslant 0$, the equation of the neutral curve will have the form (3.5).

3°. If $p_0^2 = |p_k|^2$ $(g_{002} = 0)$, and the coefficient accompanying the third power of the parameter β is not zero, then the equation of the neutral curve is obtained from a cubic equation.

Let us consider, as an example, the function $\Phi = 1 + \mu\cos\omega\tau$, in which case $p_0 = 1$, $p_1 = p_{-1} = \mu/2$, $p_k = 0$, $k = 2, 3, \ldots$. Near the principal resonance $(k = 1)$ when $|\mu| < 2$ and $\Delta < 0$ (case 2°) we have stability for small $\alpha, \varepsilon, \beta$, if on the other hand $|\mu| < 2$ and $\Delta \geqslant 0$, then the equation of the neutral curve has the form (3.5).

In the case of higher-order resonances $(k = 2, 3, \ldots)$ we always have $\Delta < 0$, and hence instability. In the case of an elastic shell we have in the same situation $(p_0 = 1, p_k = 0, k = 2, 3, \ldots)$, the condition $\Delta = 0$ holds and the neutral curve is given by the equation

$$\beta_{1,2} \approx 2\pi\alpha/\lambda^2, \quad k = 2, 3, \ldots$$

The author thanks V.I. Yudovich for his interest and for useful discussions.

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Translated by L.K.

PMM U.S.S.R., Vol.52, No.3, pp.408-412, 1988
Printed in Great Britain
0021-8928/88 $10.00+0.00

A METHOD OF ANALYSING PLATES AND SHALLOW SHELLS*

A.I. POLUBARINOVA

A method of representing a function of two variables defined in a square $\sigma = [0, \pi] \times [0, \pi]$, in the form of a combination of polynomials and differentiable trigonometric series is given. Unlike the representations obtained earlier /1-3/, the present paper proposes the use of expansions in trigonometric series over the system of functions $\{\sin mx\}$, $\{1, \cos mx\}$, $m = 1, 2, \ldots$ complete in $[0, \pi]$, and in double series over the system of functions $\{\sin mx \sin ny\}$, $\{\sin ny, \cos mx \sin ny\}$, $\{\sin mx, \sin mx \cos ny\}$, $m, n = 1, 2, \ldots$ complete in σ. Expansion in such systems of functions has certain advantages compared with expansions in the usual trigonometric system of sines and cosines in $[-\pi, \pi]$ and the corresponding system of functions in the square $[-\pi, \pi] \times [-\pi, \pi]$. The proposed method is used to solve problems of the theory of shells with constant coefficients in the case of rigid clamping along a rectangular contour. The solution is obtained in the form of trigonometric series whose coefficients are expressed in terms of the solution of an infinite linear algebraic system of equations. Numerical values of the deflection are obtained for the case of a shallow circular cylindrical shell.

**Prikl.Matem.Mekhan.*, 52, 3, 523-527, 1988